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The “switch-on” problem for linear time-invariant operators

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Abstract

The model of a linear system with stochastic input is a fundamental building block of physics. For such systems, Fourier analysis permits compact and intuitively appealing solutions. Here we extend the Fourier-based solution to include the “switch-on” problem, in which the forcing is suddenly turned on at a certain time. The modification of the solution due to a finite forcing history resembles the signal processing problem of estimating a spectrum from a finite sample. The asymptotic response of the system may then be found by considering the limiting behavior of familiar frequency-domain operators, an approach which has some advantages over the usual contour-integration method. The simple harmonic oscillator is studied in particular. The instantaneous variance is found to be proportional to the periodogram at the natural frequencies, and recovers the asymptotic behavior reported by Hasselmann (J. Fluid Mech. 12 (1962) 481–500) as the periodogram converges to the true spectrum. A simple approximation is presented which isolates the “resonant” growth of variance, and allows an useful classification according to whether this growth is linear or quadratic. Application to the physical problem of nonlinear wave –wave interaction is discussed.

Keywords: LTI operator; Harmonic oscillator; Resonance; Kernel functions

1. Introduction

Linear time-invariant systems are conveniently solved in the Fourier domain, where the time-domain convolution equation becomes a simple algebraic problem. However, when the input is random—hence not in general square-integrable—singularities may arise which render aspects of the solution ill-defined. One resolution to this difficulty is to introduce a “damping” into the system, and then examine rates of change in the limit as the damping approaches zero. Here we take an alternative approach in which a “truncated” version of the system is considered, forced only for a finite time, and then the forcing duration is allowed to approach infinity. We refer to this representation as the “moving endpoint” solution.

As an example we study in particular the simplest of physical systems, the undamped simple harmonic oscillator. Its behavior for random forcing is considered by Batchelor [2] as a preliminary to studying the much more difficult problem of homogeneous turbulence, as well as by Hasselmann [3] in the context of weakly nonlinear ocean surface wave interactions. The latter paper presents several elegant results for the asymptotic behavior of the simple harmonic oscillator; however, these are presented tersely and without proof,

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leaving open questions as to their meaning and generalization. Such asymptotic results are central to wave–wave interaction theory, and as such deserve careful attention. In particular we are interested in knowing how long one must wait before the asymptotic results become approximately true, and also how the asymptotic results relate to the instantaneous behavior of the system.

Our approach is systematic and begins with a review of Fourier theory and the theory of linear time-invariant (LTI) filters. While much of this material is standard, our presentation involves several novel features. One innovation is a compact "operator notation" which allows us to abstract the operations implied by common integrals into a symbolic form. Further, we make extensive use of the Hilbert transform, permitting complex integrations to be performed by referring to a few simple properties. Most importantly, we pay particular attention to the kernel functions which emerge as being central to this analysis.

Standard Fourier theory works exceptionally well when the forcing consists of discrete modes with independent random amplitudes. However, if the forcing is spectrally continuous and is allowed to continue indefinitely after an initial "switch-on" moment, singularities may arise which render aspects of the solution undefined. Further, a clear definition of what is meant by "resonant behavior", and an approximation which isolates such behavior, is no longer obvious.

A solution for the switch-on problem of a stochastic oscillator is obtained which we term the "moving endpoint" solution. This solution has no singularities for finite forcing duration, and consequently it may be used to find the asymptotic behavior of the system. Furthermore, the moving endpoint solution is identical in form to the solution for the corresponding LTI problem in which the same system is forced for an infinite period of time. Therefore it represents a direct generalization of the Fourier theory to the non-LTI switch-on problem, with this transformation being accomplished through the substitution of a time-dependent "transfer kernel" in the place of the usual transfer function. The transfer kernel, and hence the full solution to the switch-on problem, may be directly computed for all times through a simple filtering of the transfer function.

Applying the moving endpoint solution to the simple harmonic oscillator emphasizes the analogy between the current problem and the familiar signal analysis problem of estimating spectral density from a finite sample (e.g., [7]). The asymptotic results of [3] are recovered, but the continuity of behavior between the asymptotic response and the instantaneous response is revealed. Furthermore, this solution not only remains valid when the forcing contains line spectra, it takes on a particularly simple form. The generalization to forcing processes whose frequency content is partitioned over a plane, as in the case of interacting wave triads, is straightforward.

The organization of the paper is as follows. Section 2 begins with a brief motivational discussion of the simple harmonic oscillator. Section 3 presents the background for the mathematical methods used. Section 4 introduces the switch-on problem for LTI systems and its general solution is found. In Section 5 we examine the simple harmonic oscillator in particular, and Section 6 contains a summary and conclusions. Since much of the paper consists of "well-known" material which is re-derived in a new way and with a new emphasis, references to other works are given only for special results that fall outside our scope. However, the central role of [5–7] in shaping the character of this work should be acknowledged.

2. A first look at the simple harmonic oscillator

As a motivation for the remainder of this study we consider the stochastically forced simple harmonic oscillator. The forcing time series, $x(t)$, has an autocorrelation function

$$s(\tau) = E\{\bar{x}(t)\, x(t+\tau)\}, \tag{1}$$

where the overbar denotes a complex conjugate and E denotes the expectation operator. The autocorrelation function is related to a quantity called the spectrum through

$$S(\omega) = \int s(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t, \tag{2}$$

$$s(t) = \frac{1}{2\pi} \int S(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega, \tag{3}$$

where the expressions on the right-hand side of (2) and (3) define the forward and inverse Fourier transforms, respectively. The notation $s(t) \Leftrightarrow S(\omega)$ is used as a shorthand to imply both (2) and (3), and may be read "$s(t)$ and $S(\omega)$ are a Fourier transform pair".

The forced simple harmonic oscillator equation

$$\frac{d^2}{dt^2} y(t) + \omega_0^2 y(t) = x(t) \tag{4}$$

may be interpreted physically as stating that the difference between the acceleration of a mass located at $y(t)$ and the restoring spring force acting on it must be equal to the applied force $x(t)$. When $x(t)$ and therefore $y(t)$ are complex-valued, (4) describes the behavior of a two-dimensional oscillator since the real and imaginary parts must balance separately. It also describes the undamped plane pendulum under the small angle approximation.

The two-dimensional simple harmonic oscillator conserves both energy and angular momentum in the absence of forcing. The rate of change of energy is

$$\frac{1}{2}\frac{d}{dt}\left(\left|\frac{dy}{dt}\right|^2 + \omega_0^2|y|^2\right) = \Re\left\{x\frac{d\bar{y}}{dt}\right\} \tag{5}$$

and, denoting $y = re^{i\theta}$, the rate of change of angular momentum is

$$\frac{d}{dt}\left(\frac{d\theta}{dt}r^2\right) = \Im\{x\bar{y}\}, \tag{6}$$

where the quantity on the right-hand side of (5) is the work done on the system, and that on the right-hand side of (6) is the torque; the symbols "$\Re$" and "$\Im$" denote taking the real and imaginary part, respectively. Thus the correlation quantities $E\{x\, d\bar{y}/dt\}$ and $E\{x\bar{y}\}$ are of particular interest. The term whose imaginary part is the expected torque is

$$E\{\bar{y}x\} = \frac{1}{4i\omega_0}[S(\omega_0) - S(-\omega_0)] + \frac{1}{2\pi}\int \frac{S(\omega)}{\omega_0^2 - \omega^2}\, d\omega \tag{7}$$

so that the rate of change of angular momentum depends only on the spectrum at $\pm\omega_0$, and vanishes for purely linear [e.g. $\cos(\omega_0 t)$] forcing. The expected rate of change of energy is equal to the real part of

$$E\left\{\frac{d\bar{y}}{dt}x\right\} = \frac{1}{4}[S(\omega_0) + S(-\omega_0)] + \frac{i}{2\pi\omega_0}\int \frac{\omega S(\omega)}{\omega_0^2 - \omega^2}\, d\omega \tag{8}$$

so that the energy increases at a constant rate proportional to the sum of the input spectrum at plus or minus the natural frequency. The rates of change of both energy and angular momentum therefore depend only on the value of the spectrum at plus or minus the natural frequency, and are proportional to each other for purely "one-sided" forcing.

Expressions (7) and (8) predict that the energy and angular momentum of the stochastically forced oscillator increase without bound, therefore we will be interested in knowing the rates of change of the system's statistics. In particular, a physically important problem is determining the asymptotic behavior of a system in which the forcing is switched on at a certain time. In a study of the weakly nonlinear interactions of ocean surface gravity waves, Hasselmann [3] reports the asymptotic result [2]

$$\lim_{t\to\infty} \frac{d}{dt}\sigma_y^2(t) = \frac{1}{4\omega_0^2}[S(\omega_0) + S(-\omega_0)], \tag{9}$$

where $\sigma_y^2 \equiv E\{|y|^2\}$ is the variance of zero-mean random process y which agrees with the rate of work done by the forcing (8) because $\frac{1}{2}\omega_0^2\sigma_y^2$ accounts for one-half of the energy of the oscillator. As mentioned in the introduction, one would like to know relationship of (9) to the instantaneous behavior, and the conditions under which it is approximately true. This result recalls the well-known behavior of the periodogram, the estimate of the spectrum formed from an ensemble of finite samples of length T

$$\widetilde{S}_T(\omega) \equiv \frac{E\{|\widetilde{X}_T(\omega)|^2\}}{T}, \tag{10}$$

$$\widetilde{X}_T(\omega) \equiv \int_0^T x(t)\, e^{-i\omega t}\, dt, \tag{11}$$

which asymptotes to the true spectrum

$$\lim_{T\to\infty} \widetilde{S}_T(\omega) = \widetilde{S}(\omega) \tag{12}$$

as the length of the sample approaches infinity. One might speculate that, since the instantaneous rate of change of variance can only depend on the information between the switch-on moment and the current time, the generalization of (9) to finite times should involve the periodogram of the forcing at plus or minus the natural frequency.

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[2] Hasselmann's version (9) has a factor of $\pi/2\omega_0^2$ rather than $1/4\omega_0^2$ owing to his placement of the factor of 2π in the forward transform rather than the inverse transform.

The most familiar behavior of the simple harmonic oscillator is due to sinusoidal forcing $x(t) = \mathrm{e}^{\mathrm{i}\omega_1 t}$. In this case (4) has a solution

$$y_1(t;\omega_1,\omega_0) = \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}\omega_1 t}}{\omega_0^2-\omega_1^2} - \dfrac{1}{2\omega_0} \\ \quad \times \left[\dfrac{\mathrm{e}^{\mathrm{i}\omega_0 t}}{\omega_0-\omega_1} - \dfrac{\mathrm{e}^{-\mathrm{i}\omega_0 t}}{\omega_0+\omega_1}\right], & \omega_1^2 \neq \omega_0^2, \\ t\,\dfrac{\mathrm{e}^{\mathrm{i}\omega_1 t}}{2\mathrm{i}\omega_1} - \dfrac{\sin(\omega_0 t)}{2\mathrm{i}\omega_1\omega_0}, & \omega_1^2 = \omega_0^2 \end{cases} \tag{13}$$

(using the notation of [3]) after applying the initial conditions $y(0) = y'(0) = 0$. The solution (13) consists of (complex-valued) sinusoids for $|\omega_1| \neq |\omega_0|$, with an amplitude increasing approaching resonance, and a linearly growing sinusoid for the resonant case $|\omega_1| = |\omega_0|$. The 90° phase shift between the nonresonant and resonant solutions implies that the work done on the spring ($\int x\,\mathrm{d}y$) will integrate to zero over a period for the former but not for the latter. For the nonresonant solution, work done during one phase of the cycle is undone during the opposite phase; consequently the response has a constant amplitude even though no dissipation is present. For the resonant solution, energy is continually being put into the system.

In the simple example of pushing a swing, one needs to push the swing out of phase with its motion in order to increase the envelope of its oscillations. Therefore one waits until after the swing has reached its maximum excursion before exerting a force. However by doing work to both decelerate and accelerate the swing, one can always cause it to oscillate with a given frequency. This difference in character between resonant and off-resonant solutions is contained in a second asymptotic result from [3]

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_y^2 = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} E\{|y_1(t;\omega_1,\omega_0)|^2\} = \frac{\pi}{2\omega_0^2}[\delta(\omega_1+\omega_0) - \delta(\omega_1-\omega_0)], \tag{14}$$

which states that the variance asymptotes to a constant unless the oscillator is forced at the resonant frequency, in which case the variance becomes infinite. Comparing with (9) one notes a higher-order singularity in the discrete-mode case than in the case of the continuous spectrum.

For a forcing with the series expansion

$$x(t) = \sum_n \alpha_n \mathrm{e}^{\mathrm{i}\omega_n t}, \tag{15}$$

where the α_n are pairwise independent random variables, the response of the simple harmonic oscillator is simply

$$y(t) \equiv \sum_n \alpha_n y_1(t;\omega_n,\omega_0) \tag{16}$$

and the asymptotic behaviors of the individual terms are known from (14). If a forcing term occurs at the natural frequency, one expects the linearly growing term to dominate after a sufficiently long time ($t \gg 2\pi/\Delta\omega$), and one may approximate the rate of change of variance as

$$\frac{\mathrm{d}\sigma_y^2}{\mathrm{d}t} \approx \frac{t}{2\omega_0^2}[\alpha_r^2 + \alpha_{-r}^2], \tag{17}$$

where $|\omega_n| = |\omega_0|$ for $n = r$; this agrees with (14) that the rate of change of variance increases without bound. The resonant phenomenon of continuous growth in energy is associated exclusively with the "resonant" term (17).

However, let us consider the case in which the system is forced for a finite time T but the stochastic forcing does not consist of discrete waves as in (15). We choose the time interval T to be a multiple of the natural period $2\pi/\omega_0$. The most straightforward method to solve for the response in this case is to project the forcing onto the Fourier frequencies $\omega_n \equiv 2\pi n/T$ (n is an integer) and interpret the α_n as the coefficients of this expansion, that is, we set

$$\alpha_n \equiv \widetilde{X}_T(\omega_n)$$

with $\widetilde{X}_T(\omega)$ defined by (11).

If the forcing duration increases to say $2T$, one must perform the Fourier decomposition again and solve again for the solution over the entire time interval. In doing so, the coefficients $\alpha_{\pm r}$ of the linear growth will change when T changes, and these linearly growing parts of the two solutions will not in general agree. Further, if T is not chosen to be a multiple of the natural period, then the "resonant" term disappears entirely. For ongoing, continuous stochastic forcing, it is evidently no longer correct to associate the continuous growth in energy with the "resonant" term (17).

Therefore the notion of a "resonant" term is grounded in the behavior of systems forced by independent discrete modes, and its generalization to continuous stochastic forcing is not clear. We would also like to understand the relationship between the asymptotic results, which state that the essence of "resonance" involves the system's behavior for long times, and the discrete-mode identification of "resonance" with individual Fourier components. In particular, it would be useful to have an approximation analagous in spirit to (17) which isolates the phenomenon of resonance in a way that is meaningful for continuous forcing.

Motivated by the issues raised in this section, we consider in a general way the response of an oscillator to an ongoing stochastic forcing that is switched on at a certain time. Our approach will be to build slowly, revisiting well-known results with a new mathematical approach and an emphasis on physical intuition. At the end of this development, we will have a compact and general formulation that trivially solves the problems of interest.

3. Methods and definition

3.1. Operator notation

The following notation will be adopted for convenience. The forward and inverse Fourier transforms will be denoted symbolically as

$$\mathscr{F}x(t)[\omega] \equiv \int x(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t = X(\omega), \tag{18}$$

$$\mathscr{F}^{-1}X(\omega)[t] \equiv \frac{1}{2\pi}\int X(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega = x(t), \tag{19}$$

where the symbols $\mathscr{F}$ and $\mathscr{F}^{-1}$ will be referred to as "operators" since they are taken to imply the indicated integrations. A set of parentheses will be used to denote the argument of an explicitly named function, such as $x(t)$, while a set of square brackets will be used to denote the argument of the output of an operator. Thus, the expression $\mathscr{F}x(t)[\omega]$ means, "the result of applying the Fourier transform to $x(t)$ is a function of ω." We refer to t in this example as the "inner" argument and ω as the "outer" argument. Lags in both the input and output can then be expressed, for example

$$\mathscr{F}x(t-t_0)[\omega-\omega_0]$$

by lagging the inner and outer arguments, respectively. If no inner argument is specified, as in

$$\mathscr{F}x[\omega] \equiv \mathscr{F}x(t)[\omega],$$

it is understood that no lag is applied to the inner argument.

An operator will be assumed to operate on whatever is to its right, up until the occurrence of a set of square brackets enclosing an independent variable such as $[\omega]$ or $[t]$, so that

$$Z(\omega) \equiv \mathscr{F}x[\omega]Y(\omega)$$

is the function of ω resulting from taking the Fourier transform of $x(t)$ and multiplying the result by $Y(\omega)$. Operators may also be combined in series as in

$$\mathscr{F}^{-1}\mathscr{F}x(t)[t] = \mathscr{F}^{-1}(\mathscr{F}x(t)[\omega])[t] = x(t).$$

From this last example it is evident that for the Fourier transform, we may write

$$\mathscr{F}^{-1}\mathscr{F} = \mathscr{F}\mathscr{F}^{-1} = 1$$

with the usual caveat that the equal sign preceding this "1" is to be interpreted in the mean square sense. Note that no special meaning is ascribed to the square brackets in the case when they merely group terms without encompassing an independent variable, for instance, the first set of brackets in the expression $[1-\mathrm{i}\mathscr{H}]X[\omega]$. Operators will be denoted by capital calligraphic letters, with the exception of the convolution operator discussed in the next subsection.

3.2. Deterministic Fourier theory

The convolution theorem states that for $G(\omega) \equiv \mathscr{F}g[\omega]$ and $H(\omega) \equiv \mathscr{F}h[\omega]$, $g(t)$ and $h(t)$ being square-integrable time series,

$$2\pi\, g(t)\, h(t) = \mathscr{F}^{-1}\{G * H[\omega]\}[t], \tag{20}$$

$$g * h[t] = \mathscr{F}^{-1}\{G(\omega)\, H(\omega)\}[t], \tag{21}$$

i.e. "convolution in the time domain is equivalent to multiplication in the frequency domain" and vice-versa. The operator "$*$"

$$g * h[t] \equiv \int g(t')\, h(t-t')\, \mathrm{d}t' \tag{22}$$

is a shorthand for convolution. The shift theorem

$$\mathscr{F}x(t-t_0)[\omega] = \mathrm{e}^{-\mathrm{i}\omega t_0}\mathscr{F}x(t)[\omega] \tag{23}$$

$$\mathscr{F}^{-1}X(\omega-\omega_0)[t] = \mathrm{e}^{\mathrm{i}\omega_0 t}\mathscr{F}^{-1}X(\omega)[t] \tag{24}$$

states that shifting in one domain is equivalent to a modulation in the other domain, and vice-versa. Finally, the "energy" of a square-integrable time series $x(t)$ may be partitioned either in the time domain or the frequency domain

$$\int |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int |X(\omega)|^2\,\mathrm{d}\omega, \tag{25}$$

a result known as Parseval's theorem.

3.3. Special functions

For certain special functions which are not square-integrable, one may define a Fourier transform in a consistent way by introducing a convergence factor and taking the limit as this factor approaches zero. The equal signs in forward (18) and inverse (19) transforms are then redefined to imply this limiting operation. In this section we will pay particular attention to the properties of certain integration kernel which will arise in the subsequent analysis.

Three such functions occurring frequently in signal analysis are the delta function

$$\delta(t-t_0) \quad\Leftrightarrow\quad \mathrm{e}^{-\mathrm{i}\omega t_0}, \tag{26}$$

$$\mathrm{e}^{\mathrm{i}\omega_0 t} \quad\Leftrightarrow\quad 2\pi\delta(\omega-\omega_0), \tag{27}$$

the unit step function

$$\Gamma(t) = \begin{cases} 0, & t<0,\\ 1/2, & t=0,\\ 1, & t>0 \end{cases} \tag{28}$$

$$\Leftrightarrow \pi\delta(\omega) - \frac{\mathrm{i}}{\omega}, \tag{29}$$

and finally

$$\mathrm{sgn}(t) = \begin{cases} -1, & t<0,\\ 0, & t=0,\\ 1, & t>0 \end{cases} \tag{30}$$

$$\Leftrightarrow -\frac{2\mathrm{i}}{\omega} \tag{31}$$

with $\mathrm{sgn}(t)$ being referred to as the signum function.

The Fourier transforms of two other basic functions, the "boxcar" and "triangle" functions, will be central to our analysis. The rectangle or "boxcar" function

$$\Pi_T(t) = \begin{cases} 1, & |t|<T/2,\\ 1/2, & |t|=T/2,\\ 0, & |t|>T/2 \end{cases} \tag{32}$$

centered on $t=0$, and a shifted version

$$\widetilde{\Pi}_T(t) = \Pi_T(t+T/2), \tag{33}$$

centered on $t=T/2$, transform to the sinc function

$$\Pi_T(t) \quad\Leftrightarrow\quad \frac{\sin(\omega T/2)}{\omega/2} \equiv 2\pi\,\mathsf{D}_T(\omega), \tag{34}$$

and a "rotated" sinc function

$$\widetilde{\Pi}_T(t) \quad\Leftrightarrow\quad \mathrm{e}^{-\mathrm{i}\omega T/2}\,\frac{\sin(\omega T/2)}{\omega/2} \equiv 2\pi\widetilde{\mathsf{D}}_T(\omega), \tag{35}$$

respectively; the function $\mathsf{D}_T(\omega)$ is referred to as the Dirichlet kernel.[3] Similarly the triangle function

$$\Lambda_T(t) = \begin{cases} |1-t/T|, & |t|<T,\\ 0, & |t|\geqslant T \end{cases} \tag{36}$$

$$= \frac{1}{T}\,\Pi_T * \Pi_T(t) \tag{37}$$

has a Fourier transform

$$\Lambda_T(t) \Leftrightarrow \frac{\sin^2(\omega T/2)}{\omega^2 T/4} = \frac{(2\pi)^2}{T}|\mathsf{D}_T(\omega)|^2$$

$$\equiv 2\pi\,\mathsf{F}_T(\omega), \tag{38}$$

where $\mathsf{F}_T(\omega)$ is known as the Fejér kernel. Using the real-valued Dirichlet kernel and the Fejér kernel, the rotated Dirichlet kernel $\widetilde{\mathsf{D}}_T(\omega)$ may be written in terms of a real and an imaginary part as

$$\widetilde{\mathsf{D}}_T(\omega) = \frac{1}{2}[\mathsf{D}_{2T}(\omega) - \mathrm{i}\omega T\mathsf{F}_T(\omega)] \tag{39}$$

$$= \frac{1}{2}\left[\mathsf{D}_{2T}(\omega) - \mathrm{i}\frac{\sin^2(\omega T/2)}{\pi\,\omega/2}\right]. \tag{40}$$

[3] D_T could be more exactly termed the continuous-time Dirichlet kernel, since the term "Dirichlet kernel" traditionally refers to a kernel similar to D_T arising when the signal has been sampled at discrete times. However, since the discrete-time version of D_T will not be used here, there is no chance of confusion. The same comment applies to the Fejér kernel introduced in (38).

From the convolution theorem, a multiplication by $\Pi_T(t)$ is equivalent to a frequency-domain convolution with $\mathsf{D}(\omega)$, and similarly for $\widetilde{\mathsf{D}}(\omega)$ and $\mathsf{F}(\omega)$ and their corresponding time-domain images. The function $\mathsf{D}(\omega)$ and its relatives are called kernels because of their occurrence in such frequency-domain integral equations. Kernel functions will be denoted by capital sans serif letters. In the following we will occasionally use the alternative notation

$$\widetilde{\mathsf{D}}(\omega, T) \equiv \widetilde{\mathsf{D}}_T(\omega)$$

to indicate that the kernel is being considered an explicit function of T, rather than T being just a parameter.

One may find the asymptotic behavior of these kernels using the theory of generalized functions (a.k.a. "distributions"). See Lighthill [4] for a detailed discussion of this theory or Appendix I of Papoulis [5] for an abbreviated treatment. The asymptotic behavior of the real-valued Dirichlet kernel is

$$\lim_{T\to\infty} \mathsf{D}_{nT}(\omega) = \delta(\omega), \tag{41}$$

where n is any positive number and $\delta(\omega)$ is the Dirac delta function. The Fejér kernel exhibits similar asymptotic behavior

$$\lim_{T\to\infty} \mathsf{F}_{nT}(\omega) = n\delta(\omega) \tag{42}$$

so that with $n = 1$ both of these kernels are equivalent definitions of the delta function in the limit as time approaches infinity. Finally the rotated Dirichlet kernel becomes

$$\lim_{T\to\infty} \widetilde{\mathsf{D}}_T(\omega) = \frac{1}{2}\left[\delta(\omega) - \frac{\mathrm{i}}{\pi\omega}\right] \tag{43}$$

$$\Leftrightarrow \frac{1}{2\pi}\,\Gamma(t) \tag{44}$$

because

$$\lim_{T\to\infty} \mathrm{i}\,\frac{\sin^2(\omega T/2)}{\pi\,\omega/2} = \lim_{T\to\infty} \frac{\mathrm{i}}{\pi\omega}[1 - \cos(\omega T)] = \frac{\mathrm{i}}{\pi\omega}$$

since the limit of $\mathrm{e}^{\mathrm{i}\omega T}$ [and consequently of $\cos(\omega T)$] vanishes in the sense of a generalized function, a result known as the Riemann–Lebesgue lemma. The limiting behavior of $\widetilde{\mathsf{D}}_T(\omega)$ is evident from the fact that the shifted boxcar function $\widetilde{\Pi}_T$ becomes the unit step function as T approaches infinity. As a shorthand for the limiting statements such as (41) the notation

$$\mathsf{D}_{nT}(\omega) \rightsquigarrow \delta(\omega) \tag{45}$$

will be used, with T understood as the limiting argument.

Additional useful properties of these three kernels may be easily verified from the definitions of the kernels themselves together with the use of trigonometric identities:

$$\mathsf{D}_T(0) = \widetilde{\mathsf{D}}_T(0) = \mathsf{F}_T(0) = T/(2\pi), \tag{46}$$

$$\mathsf{D}_T(2\omega) = \cos(\omega T/2)\ \mathsf{D}_T(\omega) = \frac{1}{2}\left(\widetilde{\mathsf{D}}_T(\omega) + \overline{\widetilde{\mathsf{D}}}_T(\omega)\right), \tag{47}$$

$$\mathsf{D}_{nT}(\omega) = n\mathsf{D}_T(n\omega), \tag{48}$$

$$\widetilde{\mathsf{D}}_{nT}(\omega) = n\widetilde{\mathsf{D}}_T(n\omega), \tag{49}$$

$$\widetilde{\mathsf{D}}_T(-\omega) = \mathrm{e}^{\mathrm{i}\omega T}\widetilde{\mathsf{D}}_T(\omega) = \overline{\widetilde{\mathsf{D}}}_T(\omega), \tag{50}$$

where the overbar denotes complex conjugation and n is any number. Further, the derivatives of these kernels have the following properties:

$$\frac{\mathrm{d}}{\mathrm{d}T}\,\mathsf{D}_T(\omega) = \frac{1}{2\pi}\cos(\omega T/2) \rightsquigarrow 0, \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}T}\widetilde{\mathsf{D}}_T(\omega) = \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega T} \rightsquigarrow 0, \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}T}\{T\mathsf{F}_T(\omega)\} = \mathsf{D}_{2T}(\omega) \rightsquigarrow \delta(\omega), \tag{53}$$

where the asymptotes to zero are a consequence of the Riemann–Lebesgue lemma.

The Dirichlet and Fejér kernels are shown in Fig. 1 as a function of both time T and frequency. For any fixed time, one has the familiar oscillating profiles of the sinc function and its square. Moving towards larger times, the central maximum of both functions tightens and increases in magnitude, while the oscillating side-lobes are continually drawn in. The behavior of these kernels at fixed frequencies, marked by the vertical lines in Fig. 1, is shown in Fig. 2. At zero frequency, both functions have linear growth. At nonzero frequencies, the Dirichlet kernel

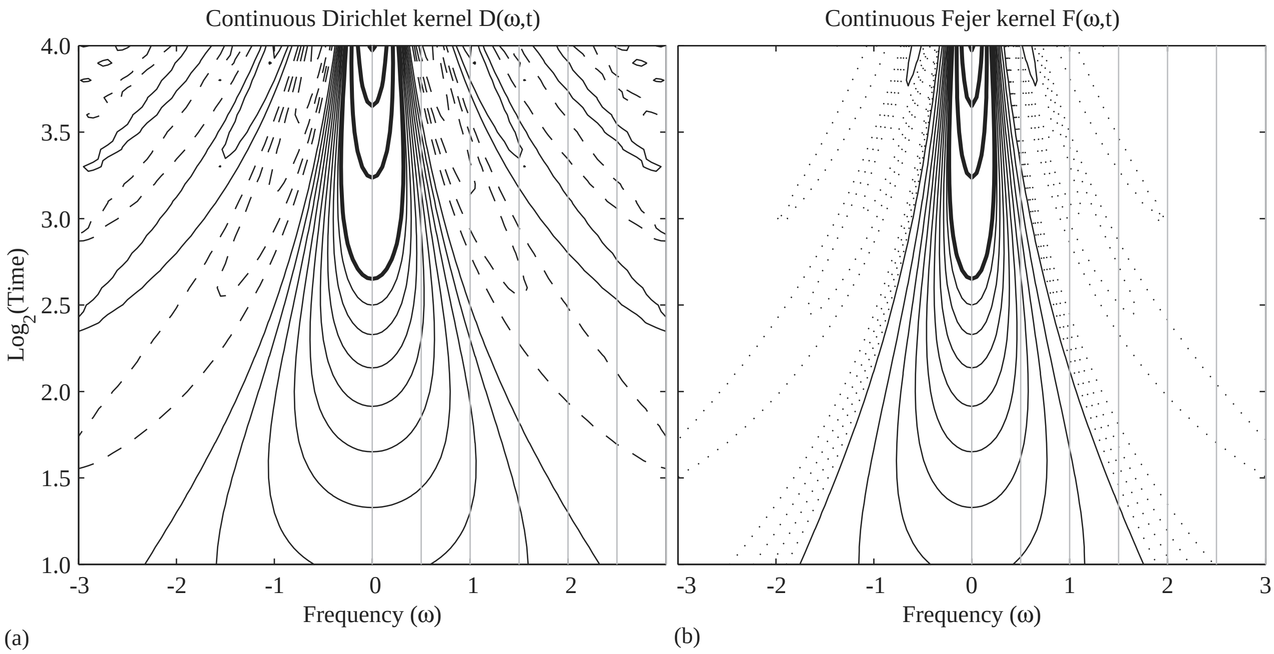

Fig. 1. The real-valued Dirichlet kernel (a) and the Fejér kernel (b) shown as a function of both time T and frequency. In (a), thin solid contours from 0.1 to 0.9 with a spacing of 0.1, dashed contours are the same but for negative values, and the thick solid contours are from 1 to 3 with a spacing of 0.5. All contours values are the same in (b) as in (a), and the additional dotted contours in (b) are from 0.02 to 0.08 with a spacing of 0.02. Vertical lines in both panels mark fixed frequencies to be referred to in the next figure.

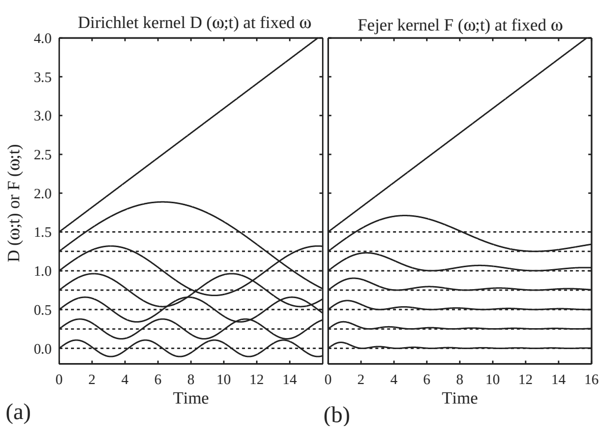

Fig. 2. The behavior as a function of time of the Dirichlet kernel (a) and the Fejér kernel (b) at fixed frequencies, (marked in Fig. 1). Curves for successively lower frequencies have been offset upwards, with the offset zero lines indicated by the dotted lines.

consists of sinusoidal oscillations everywhere, while the Fejér kernel both oscillates and decays. This illustrates, incidentally, the fact that the Fejér kernel is a more efficient (faster-converging) representation of the delta-function.

Additionally, we will need the nth order derivative of the delta function, denoted $\delta^{(n)}$, which has the property

$$\int \delta^{(n)}(\omega) f(\omega)\, \mathrm{d}\omega = (-1)^n f^{(n)}(0), \tag{54}$$

(see e.g. [5], Appendix I) and which transforms to

$$\delta^{(n)}(\omega) \quad \Leftrightarrow \quad \frac{(-\mathrm{i})^n}{2\pi}\, t^n \tag{55}$$

as may be seen by applying (54) to the inverse Fourier integral.

3.4. Stochastic Fourier theory

When $x(t)$ is a real-valued stationary stochastic time series, its time-integrated energy will be generally be infinite so its Fourier transform in the usual sense does not exist. This difficulty is solved by appealing to the randomness of $x(t)$. The equality in the inverse Fourier transform (19) is then interpreted in the mean square sense

$$\lim_{\Delta t_n \to 0} E\left\{\left|X(\omega) - \sum_n x(t_n)\,\mathrm{e}^{-\mathrm{i}\omega t_n}\,\Delta t_n\right|^2\right\} = 0$$

(see e.g. [6], Appendix 10A) that is, the expected squared difference between the right- and left-hand sides of the discrete approximation to the integral (19) vanishes as the sum formally becomes an integral. Thus the equal signs in the forward and inverse transforms have three different meanings depending on whether $x(t)$ is deterministic and of finite energy, deterministic and of infinite energy, or stochastic.

The Fourier transform $X(\omega)$ of a random process $x(t)$ is also random, and will be referred to as the "spectral process"[4] corresponding to $x(t)$. The variance of $X(\omega)$ is, in general, infinite; specifically, for bandlimited $X(\omega)$ Parseval's theorem implies $X(\omega)$ must have infinite variance over some range of frequencies in order to support the infinite energy of $x(t)$. One may show that the covariance of the spectral process $X(\omega)$ is

$$2\pi S(\omega_1)\delta(\omega_1 - \omega_2) = E\{X(\omega_1)\overline{X}(\omega_2)\} \tag{56}$$

(using (18), (1), (2), and (27) or see [6, Section 12.4]) which is zero everywhere except along the diagonal $\omega_1 = \omega_2$. Alternatively, one may use the variance of $X(\omega)$ to define the spectrum $S(\omega)$ by writing (56), and then work backwards to show that $S(\omega)$ so defined is the Fourier transform of the autocorrelation function (2).

If one has an observation of $x(t)$ from $-T/2$ to $T/2$, then the "apparent" spectral process is

$$X_T(\omega) \equiv \int \mathsf{D}_T(\omega - \omega')\,X(\omega')\,\mathrm{d}\omega' \tag{57}$$

$$\Leftrightarrow \Pi_T(t)\,x(t), \tag{58}$$

assuming $x(t)$ to be zero outside of the observation window. The apparent spectral process is used to form the periodogram estimator of the spectrum

$$S_T(\omega) \equiv \frac{E\{|X_T(\omega)|^2\}}{T}$$

$$= \int \mathsf{F}_T(\omega - \omega')\,S(\omega')\,\mathrm{d}\omega', \tag{59}$$

which asymptotes to the true spectrum

$$\lim_{T\to\infty} S_T(\omega) = S(\omega) \tag{60}$$

as a consequence of the asymptotic behavior of the Fejér kernel (42).[5]

Note from (59) that the variance of the apparent spectral process grows with T. If the true spectrum is white then the variance of the apparent process is just a constant times T. In this case the apparent spectral process

$$X_T(\omega) = \int_{-T/2}^{T/2} x(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{61}$$

may be broken up, by considering small time intervals, into contributions from independent random variables of identical variance. It is then clear that the growth of variance for a white spectrum is equivalent to the growth of variance in the random walk problem, i.e. $y_N \equiv \sum_{n=1}^{N} x_n$ implies $\sigma^2_{y_N} = N\sigma^2_x$ for identically distributed random variables x_n. Since any continuous spectrum will appear locally white over a sufficiently small frequency interval, one expects the apparent spectral process will asymptotically exhibit such random-walk behavior. This is easily verified since

$$\lim_{T\to\infty} \frac{\mathrm{d}}{\mathrm{d}T} E\{|X_T(\omega)|^2\} = S(\omega) \tag{62}$$

using (53) and (59), i.e. for a continuous spectrum one has the asymptotic results that the variance of the apparent spectral process grows linearly with the length of the record.

The covariance of the apparent spectral process $X_T(\omega)$ is

$$E\{X_T(\omega_1)\overline{X}_T(\omega_2)\} = 2\pi \times \int \mathsf{D}_T(\omega' - \omega_1)\,\mathsf{D}_T(\omega' - \omega_2) S(\omega')\,\mathrm{d}\omega', \tag{63}$$

[4] This choice of terminology is not standard. $X(\omega)$ should not be confused with the orthogonal-increment process $\mathrm{d}X^I(\omega)$ arising in the Riemann–Stieltjes formulation $x(t) = (2\pi)^{-1}\int \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}X^I(\omega)$. $X(\omega)$ and $\mathrm{d}X^I(\omega)$ are related through $X(\omega)\,\mathrm{d}\omega = \mathrm{d}X^I(\omega)$. We choose to work with $X(\omega)$ rather than the Riemann–Stieltjes formulation as a matter of preference.

[5] In order for such asymptotic results to hold, S must be "well behaved" in a sense specified in [4] or Appendix I of [5].

which vanishes in the limit as T approaches infinity for $\omega_1 \neq \omega_2$ on account of the misalignment of the two Dirichlet kernels. On the other hand for $\omega_1 = \omega_2$ we have (63) equals $TS_T(\omega)$. One may easily show that

$$\lim_{T\to\infty} E\{X_T(\omega_1)\overline{X}_T(\omega_2)\} = 2\pi\, S(\omega_1)\,\delta(\omega_1 - \omega_2) \tag{64}$$

using (63), (41), the product rule for limits, and the definition of the delta-function. Thus the covariance of the apparent process $X_T(\omega)$ asymptotes to the covariance of the true process (56).

A useful special case is a time series consisting of discrete modes having random amplitudes

$$x(t) = \sum_n \alpha_n\, e^{i\omega_n t} \Leftrightarrow X(\omega) = 2\pi \sum_n \alpha_n\, \delta(\omega - \omega_n). \tag{65}$$

In this case, one may show from the above that

$$S(\omega) = 2\pi \sum_n |\alpha_n|^2 \delta(\omega - \omega_n).$$

If we approximate a portion of a stocastic time series over some fixed interval $[T, T]/2$ in terms of the model (65), the coefficients of the expansion are given by $\alpha = X_T(\omega_n)$, and the above equation shows that the coefficients at different Fourier coefficients are in general correlated with each other.

3.5. The Hilbert transform

Since in the analysis of linear operators one frequently encounters multiplications by $\operatorname{sgn}(t)$ ($\Leftrightarrow -2i/\omega$), it is convenient to introduce the Hilbert transform

$$\mathscr{H}X(\omega)[\omega] \equiv \frac{1}{\pi}\int \frac{X(\omega')}{\omega - \omega'}\, d\omega' \tag{66}$$

$$\Leftrightarrow i\,\operatorname{sgn}(t)\,x(t) \tag{67}$$

for manipulations in the frequency domain. The fact that the Hilbert transform, and a multiplication by $i\,\operatorname{sgn}(t)$, constitute equivalent operations in two different function spaces may be compactly expressed by

$$\mathscr{F}^{-1}\mathscr{H}\mathscr{F}\, x[t] = i\,\operatorname{sgn}(t)\,x(t) \tag{68}$$

so that one may write, symbolically,

$$\mathscr{F}^{-1}\mathscr{H}\mathscr{F} = i\,\operatorname{sgn}(t) \tag{69}$$

or equivalently

$$\mathscr{F}^{-1}\mathscr{H} = i\,\operatorname{sgn}(t)\,\mathscr{F}^{-1}. \tag{70}$$

The latter "rearrangement" property means that the Hilbert transform operation may be passed through the inverse Fourier transform, a result which proves to be very useful. Note also that since

$$\mathscr{H}X(\omega - \omega_0)[\omega] = \mathscr{H}X(\omega)[\omega - \omega_0],$$

the Hilbert transform is a shift-invariant filter. The Hilbert transform has the useful, and rather mysterious, property that

$$-\mathscr{H}\mathscr{H}X[\omega] = X(\omega) \tag{71}$$

as can be seen at once by considering the equivalent time-domain statement

$$\operatorname{sgn}(t)\,\operatorname{sgn}(t)\,x(t) = x(t) \tag{72}$$

which becomes (71) after application of the convolution theorem. Thus symbolically we have

$$-\mathscr{H}\mathscr{H} = 1 \tag{73}$$

so that the inverse operator to the Hilbert transform is just the negative of a second Hilbert transform.

Several other useful properties follow at once from (71)

$$\mathscr{H}[1 - i\mathscr{H}] = i[1 - i\mathscr{H}], \tag{74}$$

$$[1 - i\mathscr{H}][1 - i\mathscr{H}] = 2[1 - i\mathscr{H}], \tag{75}$$

$$[1 - i\mathscr{H}][1 + i\mathscr{H}] = 0, \tag{76}$$

which are the frequency domain equivalent of the obvious time-domain statements

$$\operatorname{sgn}(t)[1 + \operatorname{sgn}(t)] = [1 + \operatorname{sgn}(t)], \tag{77}$$

$$[1 + \operatorname{sgn}(t)][1 + \operatorname{sgn}(t)] = 2[1 + \operatorname{sgn}(t)], \tag{78}$$

$$[1 + \operatorname{sgn}(t)][1 - \operatorname{sgn}(t)] = 0. \tag{79}$$

From (75), we observe that $[1 \pm i\mathscr{H}]/2$ are projection operators, that is, multiple applications of these operators recover the original operator. Similarly, multiplication by a shifted signum function

$$\mathrm{i}\,\mathrm{sgn}(t-T) \;\Leftrightarrow\; \mathrm{e}^{-\mathrm{i}\omega T}\mathscr{H}\mathrm{e}^{\mathrm{i}\omega T} \tag{80}$$

is accomplished by an operator which modulates, takes the Hilbert transform, and demodulates; i.e. in the time domain, we shift, multiply by $\mathrm{sgn}(t)$, and shift back.

The Hilbert transform converts delta functions $\delta(\omega)$ and simple poles $1/\omega$ into each other

$$\mathscr{H}\,\delta[\omega] = (\pi\omega)^{-1}, \tag{81}$$

$$\mathscr{H}\,\omega^{-1}[\omega] = -\pi\,\delta(\omega) \tag{82}$$

from which it follows that

$$\int \mathscr{H}\,\delta[\omega-\omega']\,X(\omega')\,\mathrm{d}\omega' = \mathscr{H}X[\omega], \tag{83}$$

in other words, convolving with the Hilbert transform of the delta function is equivalent to performing the Hilbert transform. These relations may be extended to derivatives of the delta-function

$$\mathscr{H}\,\delta^{(n)}[\omega] = \frac{n!}{\pi}\,\frac{(-1)^n}{\omega^{n+1}}, \tag{84}$$

$$\mathscr{H}\,\omega^{-n}[\omega] = \frac{\pi}{(n-1)!}\,(-1)^n\,\delta^{(n-1)}(\omega), \tag{85}$$

so that the Hilbert transform converts nth-order derivatives of the delta function into $(n+1)$th order poles and vice versa.

Some of the special functions introduced in Section 3.3 may be represented more compactly in terms of the Hilbert transform. The Fourier transform of the unit step function is

$$\Gamma(t) \;\Leftrightarrow\; \pi\delta(\omega) - \frac{\mathrm{i}}{\omega} \tag{86}$$

$$\Leftrightarrow \pi[1-\mathrm{i}\mathscr{H}]\,\delta(\omega), \tag{87}$$

while the rotated Dirichlet kernel may be decomposed as

$$\widetilde{\mathsf{D}}_T(\omega) = \frac{1}{2}\,[\mathsf{D}_{2T}(\omega) - \mathrm{i}\omega T\mathsf{F}_T(\omega)]$$

$$= \frac{1}{2}[1-\mathrm{i}\mathscr{H}]\mathsf{D}_{2T}[\omega], \tag{88}$$

the real part being a simple Dirichlet kernel and the imaginary part being the Hilbert transform of that kernel. It follows that

$$\mathscr{H}\widetilde{\mathsf{D}}_T(\omega) = \mathrm{i}\widetilde{\mathsf{D}}_T(\omega)$$

using property (74). The asymptotic behavior of the rotated Dirichlet kernel, $\widetilde{\mathsf{D}}_T(\omega) \rightsquigarrow (2\pi)^{-1}\mathscr{F}\Gamma[\omega]$, may be seen at once by comparing (88) and (87). Also, various useful Fourier transforms may be derived by the properties of the Hilbert transform, for example

$$\mathscr{F}^{-1}\omega^{-n}[t]$$

$$= \mathscr{F}^{-1}\mathscr{H}\left\{\frac{\pi}{(n-1)!}\,(-1)^{n-1}\delta^{(n-1)}(\omega)\right\}[t]$$

$$= \mathrm{i}\,\mathrm{sgn}(t)\,\mathscr{F}^{-1}\left\{\frac{\pi}{(n-1)!}\,(-1)^{n-1}\delta^{(n-1)}(\omega)\right\}[t]$$

$$= \frac{\mathrm{i}^n}{2(n-1)!}\,\mathrm{sgn}(t)\,t^{n-1} \tag{89}$$

using (84), (70), and (55).

The algebra of the following sections is vastly simplified through the use of the Hilbert transform.

4. Linear operators

4.1. Time-invariant operators

We write the equation for the simple harmonic oscillator (4) symbolically in terms of the differential operator $\mathscr{L}_0^{-1}$

$$\mathscr{L}_0^{-1}\,y[t] = x(t) \tag{90}$$

with

$$\mathscr{L}_0^{-1} \equiv \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_0^2\right). \tag{91}$$

This "inverse" operator $\mathscr{L}_0^{-1}$ determines the forcing $x(t)$ which would be necessary to create a certain response $y(t)$, that is, it maps the response onto the forcing. More useful is the "forwards" operator $\mathscr{L}_0$

$$y(t) = \mathscr{L}_0\,x[t], \tag{92}$$

which determines the response from a prescribed forcing. Obviously, $\mathscr{L}_0$ depends upon the initial conditions while $\mathscr{L}_0^{-1}$ does not. These two operators are symbolic inverses of one another since

$$x(t) = \mathscr{L}_0^{-1}\mathscr{L}_0\,x[t], \tag{93}$$

$$y(t) = \mathscr{L}_0\mathscr{L}_0^{-1}\,y[t] \tag{94}$$

and thus we may write $\mathscr{L}_0\mathscr{L}_0^{-1} = \mathscr{L}_0^{-1}\mathscr{L}_0 = 1$.

Consider now a more general operator $\mathscr{L}$, assumed to have the properties of linearity

$$\mathscr{L}\{c_1 x_1(t)\, c_2 x_2(t)\}[t] = c_1 \mathscr{L} x_1[t] + c_2 \mathscr{L} x_2[t] \tag{95}$$

and shift- or time-invariance

$$\mathscr{L} x(t - t_0)[t] = \mathscr{L} x(t)[t - t_0]. \tag{96}$$

It follows from these two properties (see e.g. [7]) that $\mathscr{L}$ may be expressed as a convolution

$$\mathscr{L}\, x[t] = g * x[t] = \int g(t - t')\, x(t')\, \mathrm{d}t', \tag{97}$$

where $g(t)$ is known as the Green's function for $\mathscr{L}$. The Green's function may be found by applying the operator $\mathscr{L}$ to the delta-function, since

$$\mathscr{L}\delta[t] = \int g(t - t')\, \delta(t')\, \mathrm{d}t' = g(t). \tag{98}$$

Furthermore, from the convolution theorem, the expression of $\mathscr{L}x[t]$ as a time-domain convolution is equivalent to

$$y(t) \equiv \mathscr{L}x[t] = g * x[t] = \mathscr{F}^{-1} G(\omega)\, X(\omega) \\ = \mathscr{F}^{-1} Y(\omega) \tag{99}$$

with $G(\omega) \equiv \mathscr{F} g[\omega]$ and $Y(\omega) \equiv \mathscr{F} y[\omega]$. Since in the frequency domain the convolution equation is replaced with a simple multiplication, it is often more convenient to solve for the response in the frequency domain and then evaluate the inverse Fourier transform implied by (99)

$$y(t) = \frac{1}{2\pi} \int G(\omega)\, X(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{100}$$

It follows from (99) that the inverse operator to $\mathscr{L}$ has a Fourier representation

$$x(t) = \mathscr{L}^{-1} y[t] \quad \Leftrightarrow \quad X(\omega) = \frac{Y(\omega)}{G(\omega)}. \tag{101}$$

In physical problems it is generally the case the differential operator $\mathscr{L}^{-1}$ is known, and one wishes to find the solution to the problem beginning with this knowledge.

The Fourier transform of $g(t)$, $G(\omega)$, is referred to as the transfer function in the signal processing literature. The transfer function may found immediately by applying the inverse (differential) operator to a sinusoid

$$\mathscr{L}^{-1} \mathrm{e}^{\mathrm{i}\omega_1 t}[t] = \frac{1}{G(\omega_1)} \mathrm{e}^{\mathrm{i}\omega_1 t} \tag{102}$$

since in the Fourier domain we have

$$\mathscr{L}^{-1} \mathrm{e}^{\mathrm{i}\omega_1 t}[t] = \mathscr{F}^{-1} \frac{2\pi\, \delta(\omega - \omega_1)}{G(\omega)}[t] \tag{103}$$

$$= \int \frac{\delta(\omega' - \omega_1)}{G(\omega')} \mathrm{e}^{\mathrm{i}\omega' t}\, \mathrm{d}\omega' \tag{104}$$

which integrates to the right-hand side of (102). Furthermore

$$\mathscr{L} \mathrm{e}^{\mathrm{i}\omega_1 t}[t] = G(\omega_1)\, \mathrm{e}^{\mathrm{i}\omega_1 t} \tag{105}$$

then follows from $\mathscr{L}\mathscr{L}^{-1} = 1$. Thus, the complex sinusoids are eigenvectors of both operators, with eigenvalues $1/G(\omega)$ for the inverse operator and eigenvalues $G(\omega)$ for the forward operator.

Those frequencies for which $1/G(\omega)$ vanishes constitute the free waves of the system, that is, sinusoids which satisfy (90) with the forcing $x(t)$ set to zero. Equivalently, (105) states that if the forcing contains discrete waves of any finite amplitude at these frequencies, the response will have infinite amplitude. However in the case of a continuous forcing spectrum $S(\omega)$ with finite spectral density everywhere, the singularities in the inverse Fourier transform (100) will in general be integrable. The application of initial conditions to the system constitutes assigning meaning to the integration over these singularities.

4.2. The simple harmonic oscillator

Returning to the simple harmonic oscillator, we use (102) to find

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_0^2\right) \mathrm{e}^{\mathrm{i}\omega_1 t}[t] = (\omega_0^2 - \omega_1^2)\, \mathrm{e}^{\mathrm{i}\omega_1 t} \tag{106}$$

and therefore

$$G_{\mathrm{a}}(\omega) = \frac{1}{\omega_0^2 - \omega^2} \\ = \frac{1}{2\omega_0} \left(\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right), \tag{107}$$

where the meaning of the subscript "a" will be given below. To find the Green's function, we evaluate the inverse transform of (107) from the inverse Fourier transform of $1/\omega$ (31) together with the shift theorem (24). Alternatively one may use (81) to write

$$g_a(t) = \frac{\pi}{2\omega_0}\mathscr{F}^{-1}\mathscr{H}\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}[t] \tag{108}$$

$$= \frac{\pi}{2\omega_0} \mathrm{i}\,\mathrm{sgn}(t)\left(\mathrm{e}^{-\mathrm{i}\omega_0 t} - \mathrm{e}^{\mathrm{i}\omega_0 t}\right) \tag{109}$$

$$= \frac{1}{2\omega_0}\mathrm{sgn}(t)\sin(\omega_0 t) \tag{110}$$

after applying the "rearrangement" property (70).

Note that since the Green's function is nonzero for both positive and negative times, the solution for a delta-function "impulse" will cause the system to "ring" for negative times, before the impulse is applied. Such a response is termed "acausal" (hence the subscript). In physical problems in which t is interpreted as time, one is usually interested in the "causal" response which vanishes before the forcing is applied.

To find the causal response for the simple harmonic oscillator, we add the acausal Green's function $g_a(t)$ to a function that will make it vanish before time $t=0$, namely itself times the signum function

$$g_c(t) = [1+\mathrm{sgn}(t)]g_a(t) \tag{111}$$

$$= g_a(t) + \frac{\sin(\omega_0 t)}{2\omega_0} \tag{112}$$

$$= \frac{\Gamma(t)}{\omega_0}\sin(\omega_0 t), \tag{113}$$

which is equivalent to adding an appropriately chosen free wave. Eq. (113) now means that when you hit the oscillator, it rings. The causal transfer function is, from the frequency-domain version of (111),

$$G_c(\omega) \equiv [1-\mathrm{i}\mathscr{H}]G_a[\omega] = \frac{\pi}{2\omega_0}[1-\mathrm{i}\mathscr{H}]\mathscr{H} \tag{114}$$

$$\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}[\omega] \tag{115}$$

$$= \frac{\pi}{2\omega_0}[\mathrm{i}+\mathscr{H}] \times\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}[\omega] \tag{116}$$

using (81) and (73). Expanding the Hilbert transform terms one has

$$G(\omega) = \frac{\pi}{2\mathrm{i}\omega_0}\{\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\} + \frac{1}{\omega_0^2-\omega^2}, \tag{117}$$

which consists of a line response at the natural frequencies $\pm\omega_0$, plus a contribution that is distributed across a range of frequencies. The line response lags the forcing by 90°, while the continuous response preserves the phase of the forcing's Fourier components. Comparing (117) with (107), it is evident that the line portion of the causal transfer function is due to the initial conditions while the continuous portion is due to the acausal response. Furthermore, it is evident that the delta-functions in the causal transfer function assign meaning to the singularities at $\pm\omega_0$.

4.3. General expressions

In the same manner as above, one can find general expressions for the Green's functions and transfer functions of a linear time-invariant operator. The acausal transfer function may be found by factoring the differential operator $\mathscr{L}^{-1}$

$$\mathscr{L}^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{i}\omega_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{i}\,\omega_2\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{i}\,\omega_N\right), \tag{118}$$

where the N roots $\mathrm{i}\omega_n$ may be real or complex and need not be distinct; note the coefficient of the highest order derivative has been set to one. Using a partial fraction expansion, one obtains the transfer function

$$G_a(\omega) = \sum_{n=1}^{D} c_n\left(\frac{\mathrm{i}}{\omega_n-\omega}\right)^{r_n}, \tag{119}$$

where the c_n are constants arising from the partial fraction expansion, r_n is the multiplicity of the nth root, and D is the number of distinct roots. The causal transfer function may then be written compactly in frequency domain using (114), which evaluates to

$$G_a(\omega) = \sum_{n=1}^{D} c_n\left(\frac{\mathrm{i}}{\omega_n-\omega}\right)^{r_n} - c_n\frac{\pi\mathrm{i}^{r_n+1}}{(r_n-1)!} \delta^{(n-1)}(\omega-\omega_n), \tag{120}$$

using (85). One finds the Green's functions are:

$$g_a(t) = \sum_{n=1}^{D}\frac{c_n}{(r_n-1)!}\frac{\mathrm{sgn}(t)}{2}\,t^{r_n-1}\,\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{121}$$

$$g_c(t) = \sum_{n=1}^{D}\frac{c_n}{(r_n-1)!}\,\Gamma(t)\,t^{r_n-1}\,\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{122}$$

using (89) together with the shift theorem (24).

Since the acausal Green's function consists of a sum of free waves multiplied by a signum function, multiplication by a second signum function recovers a free wave. This may be verified directly by inserting $\mathrm{sgn}(t)g_a(t)$ back into the original differential equation. For example, for the special case

$$\mathscr{L}^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\omega_0\right)^n, \tag{123}$$

one has

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\omega_0\right)^{n-1} t^{n-1}\mathrm{e}^{\mathrm{i}\omega_0 t} = (n-1)! \times \mathrm{e}^{\mathrm{i}\omega_0 t}, \tag{124}$$

which vanishes upon the nth and final derivative; the case for operators consisting of multiple poles with distinct roots follows trivially. It is because $\mathrm{sgn}(t)g_\mathrm{a}(t)$ is a free wave that one may simply write (111) to create the causal Green's function, and consequently (114) for the causal transfer function.

4.4. Stochastic forcing

When the forcing $x(t)$ is a square-integrable deterministic function, all Fourier transforms are well defined and the problem may be solved by evaluating (100) once the causal transfer function is known. When $x(t)$ is a stationary stochastic time series with spectrum $S(\omega)$, Fourier transforms such as (100) are still well-defined provided they are assigned a new meaning as discussed in Section 3.4. However, in this case the problem is not yet solved, since one is interested in the relationship between the statistics of the forcing and those of the response.

For a stochastically-forced linear time-invariant operator with transfer function $G(\omega)$, the lowest-order time-domain statistics at a fixed time are

$$E\{\bar{y}x\} = \frac{1}{2\pi}\int \overline{G}(\omega)\,S(\omega)\,\mathrm{d}\omega, \tag{125}$$

$$\sigma_y^2 \equiv E\{\bar{y}y\} = \frac{1}{2\pi}\int |G(\omega)|^2 S(\omega)\,\mathrm{d}\omega, \tag{126}$$

$$E\left\{\frac{\mathrm{d}\bar{y}}{\mathrm{d}t}x\right\} = \frac{-\mathrm{i}}{2\pi}\int \omega\overline{G}(\omega)\,S(\omega)\,\mathrm{d}\omega, \tag{127}$$

$$E\left\{\frac{\mathrm{d}\bar{y}}{\mathrm{d}t}y\right\} = \frac{-\mathrm{i}}{2\pi}\int \omega|G(\omega)|^2 S(\omega)\,\mathrm{d}\omega, \tag{128}$$

as may be verified by substituting (19) and (100) into the expressions on the left-hand sides and using (56). The transfer function in the above may be the causal or acausal transfer function, as desired.

As was seen in the case of the simple harmonic oscillator, the causal transfer function $G(\omega)$ will generally contain delta-functions at the free waves of the system. The occurrence of these delta-functions will limit the usefulness of (125–128). The $|G(\omega)|^2$ term in (126) will then contain squares of delta functions which integrate to infinity. Similarly when the forcing also contains a delta-function at one of the natural frequencies, the interaction of the two delta functions creates infinite terms in all these expression. In this case it is necessary to consider the limiting behavior of rates of change of these statistics. Henceforth we will only be concerned with causal systems, since this is the more physically realistic case.

4.5. A "switch-on" operator

Consider the "switch-on" problem in which the forcing is assumed to be zero before time $t=0$. Define a new operator and its associated output

$$y_\gamma(t) \equiv \mathscr{L}_\gamma x[t] \equiv \mathscr{L}\Gamma(t)x(t)[t] = \frac{1}{2}\mathscr{L}\{1+\mathrm{sgn}(t)\}x(t)[t]. \tag{129}$$

The solution to $y_\gamma(t)$ may be found by directly integrating the Green's function over half the time axis, or equivalently from the inverse Fourier transform (100) with $X(\omega)$ replaced by the "smoothed" version

$$X_\gamma(\omega) \equiv \frac{1}{2}[1-\mathrm{i}\mathscr{H}]X[\omega]. \tag{130}$$

Note, however, that the smoothing operation introduces correlations across frequencies, i.e. $E\{X_\gamma(\omega_1)\overline{X}_\gamma(\omega_2)\} \neq 0$ for $\omega_1 \neq \omega_2$.

It is convenient to express the statistics of the system, which will necessarily be time-dependent, in terms of the spectrum of the forcing process. Interchanging the orders of the $1/2[1-\mathrm{i}\mathscr{H}]$ operator and the inverse Fourier transform in (100), one obtains

$$y_\gamma(t) = \frac{1}{2\pi}\int \mathsf{G}_\gamma(\omega,t)X(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{131}$$

where the time-dependent kernel $\mathsf{G}_\gamma(\omega, t)$ is defined by

$$\mathsf{G}_\gamma(\omega,t) \equiv \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega t}[1+\mathrm{i}\mathscr{H}]\,\mathrm{e}^{\mathrm{i}\omega t}G_\mathrm{c}(\omega)[\omega]$$
$$\equiv \frac{1}{2}\,G_\mathrm{c}(\omega) + \mathrm{i}\,\frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\int \frac{\mathrm{e}^{\mathrm{i}\omega' t}G_\mathrm{c}(\omega')}{\omega-\omega'}\,\mathrm{d}\omega'.$$

Note that since the switch-on operation is not time-invariant, (131) is no longer an inverse Fourier transform. To determine the statistics of system for all time, one may now simply replace $G(\omega)$ in (125–128) with the transfer kernel $\mathsf{G}_\gamma(\omega, t)$. However, the aforementioned problems with delta functions in $G_\mathrm{c}(\omega)$ remain, since $G_\mathrm{c}(\omega)$ explicitly appears in $\mathsf{G}_\gamma(\omega, t)$. This occurs because the solution to the switch-on problem is defined with reference to the solution for all time, as is evident from (129).

One solution to this difficulty is to introduce a fictitious "damping" into the oscillator's Green's function, and then integrate (125–128) or their derivatives on the complex plane using contour integration to isolate the contributions from the singularities, and finally taking the limit as the damping parameter approaches zero. Instead we will take another approach by expressing the solution directly in terms of kernel functions whose asymptotic properties have already been established.

4.6. The "moving endpoint" solution

To avoid the occurrence of delta-functions in the time-dependent transfer kernel, we construct a solution in which any delta-functions in the causal transfer function are "smoothed" away for finite times. Consider a system in which both input and output are set to zero outside a specified time window

$$y_T(t) \equiv \mathscr{L}_T\,x[t] \equiv \widetilde{\Pi}_T(t)\,\mathscr{L}\widetilde{\Pi}_T x[t]. \qquad (132)$$

This consists of forcing the original system $\mathscr{L}$ with $x(t)$ over the time interval $t = 0$ to $t = T$, then truncating the output to remove "ringing" of the system (i.e. homogeneous solutions) after the forcing is turned off. This is obviously not a time-invariant filter, so we do not expect the solution to be given by a simple multiplication in the frequency domain; rather, the time-domain multiplications are expected to introduce frequency-domain convolutions.

A truncated version of the forcing is defined by

$$\tilde{x}_T(t) \equiv \widetilde{\Pi}_T(t)\,x(t) \qquad (133)$$

with Fourier transform

$$\widetilde{X}_T(\omega) \equiv \int \widetilde{\mathsf{D}}_T(\omega-\omega')\,X(\omega')\,\mathrm{d}\omega'. \qquad (134)$$

The response of the operator therefore depends not upon the true spectral process $X(\omega)$, but only the "apparent" process $\widetilde{X}_T(\omega)$ based on the amount of information available over the duration of the forcing. As T approaches infinity, $\widetilde{X}_T(\omega)$ approaches $\widetilde{X}_\gamma(\omega)$—the forcing for the switch-on problem—as expected [see property (44)].

Note that here $x(t)$ has been truncated outside of the interval $(0, T)$ so that the apparent process $\widetilde{X}_T(\omega)$ is the convolution of $X(\omega)$ with the rotated (complex-valued) Dirichlet kernel. By contrast in Section 3.4 we truncated $x(t)$ outside of $(-T, T)/2$, and the apparent process $X_T(\omega)$ involved a convolution with an ordinary (real-valued) Dirichlet kernel. The apparent spectra in the two cases are the same, however, and as in (59) result from convolving the true spectrum with the Fejér kernel.

It is convenient to write the solution in such a way that the true process $X(\omega)$ is multiplied by some time-varying version of the original causal transfer function $G(\omega)$. Substituting (134) into (100) and rearranging the order of integration yields an equivalent expression for $y_T(t)$ in terms of a time-dependent transfer kernel

$$y_T(t) = \widetilde{\Pi}_T(t)\,\frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega'' t}\times$$
$$\left[\int G_\mathrm{c}(\omega')\mathsf{D}_T(\omega'-\omega'')\,\mathrm{e}^{\mathrm{i}(\omega'-\omega'')t}\,\mathrm{d}\omega'\right]$$
$$X(\omega')\,\mathrm{d}\omega' \qquad (135)$$

which is the term in brackets. Choosing $t = T$, the end of the forcing interval, the solution becomes

$$y_T(T) = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega T}\widetilde{\mathsf{G}}(\omega, T)\,X(\omega')\,\mathrm{d}\omega', \qquad (136)$$

where

$$\widetilde{\mathsf{G}}(\omega, T) \equiv \int G_\mathrm{c}(\omega')\widetilde{\mathsf{D}}(\omega-\omega', T)\,\mathrm{d}\omega' \qquad (137)$$

defines a time-varying "transfer kernel" which at any fixed time T is just a smoothed version of the transfer function.

Notice that since (136) is valid for all $T>0$, we have a solution to the "switch-on" problem

$$y_\gamma(t) = \Gamma(t)\,\frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega t}\widetilde{\mathsf{G}}(\omega,t)\,X(\omega)\,\mathrm{d}\omega, \tag{138}$$

which is valid even if both the original causal transfer function $G_c(\omega)$ and the forcing process $X(\omega)$ contain delta functions. Further, as t goes to infinity, $\widetilde{\mathsf{G}}(\omega,t)$ approaches $G_c(\omega)$ as expected, since

$$\lim_{t\to\infty}\widetilde{\mathsf{G}}(\omega,t) = \frac{1}{2}[1-\mathrm{i}\mathscr{H}]G_c[\omega] \tag{139}$$

$$= \frac{1}{2}[1-\mathrm{i}\mathscr{H}][1-\mathrm{i}\mathscr{H}]G_a[\omega] \tag{140}$$

$$= [1-\mathrm{i}\mathscr{H}]G_a[\omega] \tag{141}$$

$$= G_c(\omega) \tag{142}$$

after invoking property (75) of the Hilbert transform between the second and the third lines.

Solution (138) will be referred to as the "moving endpoint" solution and is valid for all linear time-invariant operators. Expressions (125–128) may now be evaluated by substituting $\widetilde{\mathsf{G}}(\omega,t)$ for $G_c(\omega)$ to find the statistics of the system at all times, thus solving the switch-on problem. Therefore, we now have expressions for the solution to the switch-on problem and its time-varying statistics which are identical in form to those for infinite-duration forcing and which asymptotically recover them. This generalizes the frequency-domain solution for the linear time-invariant operator $\mathscr{L}$ to the non-time-invariant switch-on operator $\mathscr{L}_\gamma$. As a consequence, note that in (138) both the complex exponential and the filter depend upon t, hence this expression may no longer be interpreted as a Fourier transform. Unlike (100), one must project $X(\omega)$ onto a different function at each moment.

For a series forcing the solution takes on a particularly simple form. With $x(t)$ given by (65), (138) becomes

$$y_\gamma(t) = \Gamma(t)\sum_n \alpha_n \widetilde{\mathsf{G}}(\omega_n,t)\,\mathrm{e}^{\mathrm{i}\omega_n t} \tag{143}$$

whether or not the frequencies ω_n are equally spaced. Thus the solution is known for all time in terms of the coefficients α_n once the transfer kernel has been computed.

5. Application to the simple harmonic oscillator

5.1. The "moving endpoint" solution

We now use the methods developed in the previous section to study the behavior of the simple harmonic oscillator. One finds the time-dependent transfer kernel for this system is [inserting (116) into (137) and using (83)]

$$\widetilde{\mathsf{G}}(\omega,t) = \frac{\mathrm{i}\pi}{\omega_0}[\widetilde{\mathsf{D}}(\omega+\omega_0,t)-\widetilde{\mathsf{D}}(\omega-\omega_0,t)] \equiv \widetilde{\mathsf{C}}(\omega,t;\omega_0), \tag{144}$$

where the notation $\widetilde{\mathsf{C}}(\omega,t;\omega_0)$ will be used to refer to this particular transfer kernel. Expanding the Hilbert transform terms one has [using (88)]

$$\widetilde{\mathsf{C}}(\omega,\omega_0,t) = \frac{\pi}{2\omega_0}[\mathrm{i}+\mathscr{H}] \times[\mathsf{D}(\omega+\omega_0,2t)-\mathsf{D}(\omega-\omega_0,2t)], \tag{145}$$

which using (43) becomes (117) in the limit as t approaches infinity.

It is worth mentioning that in deriving (145) exactly $1/2\widetilde{\mathsf{C}}(\omega,t)$ is due to the delta functions in $G(\omega)$ interacting with the real-valued $\mathsf{D}(\omega\pm\omega_0,t)$ terms, and the other half is due to the interactions between the two Hilbert transforms (the cross terms cancel). This means that, while the "local" and "distributed" terms in the transfer function (117) may be attributed to the homogeneous and inhomogeneous responses, respectively, the same cannot be said for the "local" (imaginary) and "distributed" (real) terms in (145). Rather, the homogeneous and inhomogeneous responses contribute identically, and half of the terms which asymptotes to delta-functions in (145) arise from each. This is a subtle but important point that highlights the utility of our approach.

The switch-on problem is now solved by inserting (144) into (138). However some insight may be gained by rearranging the complex exponentials in the resulting expression to yield

$$y_\gamma(t) = \frac{\Gamma(t)}{2\mathrm{i}\omega_0}[\mathrm{e}^{\mathrm{i}\omega_0 t}\widetilde{X}(\omega_0;t)-\mathrm{e}^{-\mathrm{i}\omega_0 t}\widetilde{X}(-\omega_0;t)], \tag{146}$$

where now $\widetilde{X}(\omega;t)$ is defined as in (134) but we allow t to vary continuously. This equation states that at each time t, the response is completely determined by the instantaneous value of the apparent spectral process at $\pm\omega_0$. Loosely speaking, one could say that the oscillator is instantaneously responding with its natural frequency to forcing which appears to be occurring at its natural frequency. If the apparent spectral process asymptotes to a constant, then the response will just consist of "ringing" at the natural frequency. Solution (146) is, of course, in the form of a "variation of parameters" solution, that is, it consists of the two homogeneous solutions each multiplied by some function of time.

5.2. Sinusoidal forcing

For sinusoidal forcing the solution to the switch-on problem is

$$y(t) = \Gamma(t)\sum_n \alpha_n \widetilde{\mathsf{C}}(\omega_n, t; \omega_0)\,\mathrm{e}^{\mathrm{i}\omega_n t} \tag{147}$$

using (143).

This equation states that we place rotated sinc functions at all the Fourier frequencies, multiply by $\mathrm{e}^{\mathrm{i}\omega_n t}$, and up the resulting projections onto plus or minus the natural frequency. The solution is thus known for all time if the expansion coefficients α_n are given. Comparing with the earlier expression (16) one finds

$$y_1(t;\omega_n,\omega_0) = \widetilde{\mathsf{C}}(\omega_n, t; \omega_0)\,\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{148}$$

however, the simple structure of y_1 as a difference of Dirichlet kernels is not at all evident from its definition (13).

Solutions (147) and (13) are identical, as one may easily verify. In particular, the linear growth at resonance is recovered by $\widetilde{\mathsf{D}}_T(0) = T/(2\pi)$. The compact form of (147) is due to the fact that the coefficients of the sinusoids have been explicitly constructed to enforce the initial conditions, that is, each term contains both the forced response at ω_n together with the free wave response at $\pm\omega_0$. This formulation has substantial advantages in determining the asymptotic behavior of the solutions, as will be demonstrated in the next section.

Expression (147) also exhibits continuous behavior as a single forcing wave approaches the resonant frequency, unlike (13) in which the resonant and non-resonant solutions appear to be separate. When the oscillator is forced by a single sine wave having frequency ω_1, the solution is

$$y(t) = \frac{1}{\omega_0}\,\frac{\omega_0\sin(\omega_1 t) - \omega_1\sin(\omega_0 t)}{\omega_0^2 - \omega_1^2}. \tag{149}$$

If $\omega_1 = \omega_0 + \Delta\omega$ then this becomes

$$y(t) = \frac{\sin(\omega_0 t)\cos(\Delta\omega t) + \cos(\omega_0 t)\sin(\Delta\omega t)}{\omega_0^2 - (\omega_0+\Delta\omega)^2} - \frac{1}{\omega_0}\,\frac{(\omega_0+\Delta\omega)\sin(\omega_0 t)}{\omega_0^2 - (\omega_0+\Delta\omega)^2} \tag{150}$$

after the application of a trigonometric identity, with limiting behavior

$$\lim_{\Delta\omega\to 0} y(t) = \frac{1}{2\omega_0}\,[\sin(\omega_0 t) - t\cos(\omega_0 t)]. \tag{151}$$

Therefore the resonant solution is due to the "beating" of the free wave with the resonant wave, in the limit as the difference between these two waves becomes a zero-frequency sinusoid, i.e. a linear trend. This continuity of behavior is explicitly captured within the Dirichlet kernels, as one may see from Fig. 1.

Return now to the familiar problem of the simple harmonic oscillator forced for a fixed time T, described in Section 2. The eigenfunctions for this case are the sinusoids at the Fourier frequencies $2\pi n/T$. These will be called the "local" eigenfunctions, as distinct from the "global" eigenfunctions for which are the sinusoids at all (continuously varying) frequencies. One solves the problem by projecting the forcing onto these local eigenfunctions. However if time T is allowed to increase to a new fixed value, the local eigenfunctions will change, and the projections onto the new eigenfunctions will sample the forcing spectral process differently. The time dependence of $\widetilde{\mathsf{C}}(\omega,t)$ reflects this changing nature of the local eigenfunctions as the forcing duration is allowed to vary continuously.

5.3. Limiting behavior

The statistical behavior of the simple harmonic oscillator can now be found in terms of its transfer kernel $\widetilde{\mathsf{C}}(\omega,t)$. Specifically, from (126) the variance is given by

$$\sigma_y^2(t) = \frac{1}{2\pi}\int |\widetilde{\mathsf{C}}(\omega,t;\omega_0)|^2 S(\omega)\,\mathrm{d}\omega \tag{152}$$

with the square of the transfer kernel equal to

$$|\widetilde{\mathsf{C}}(\omega,t;\omega_0)|^2 = \frac{\pi}{2\omega_0^2}[t\,\mathsf{F}(\omega+\omega_0,t) + t\,\mathsf{F}(\omega-\omega_0,t) - \mathsf{H}(\omega,t;\omega_0)], \tag{153}$$

where we have defined a new function

$$\mathsf{H}(\omega,t;\omega_0) \equiv 4\pi\Re\left\{\widetilde{\mathsf{D}}(\omega+\omega_0,t)\overline{\widetilde{\mathsf{D}}}(\omega-\omega_0,t)\right\} \tag{154}$$

with limiting behavior

$$\lim_{t\to\infty}\mathsf{H}(\omega,t;\omega_0) = \pi\,\delta(\omega-\omega_0)\delta(\omega+\omega_0) + \frac{1}{\pi}\frac{1}{\omega^2-\omega_0^2}. \tag{155}$$

In (153) the Fejér kernels arise from the sinc functions at the same poles "seeing" each other, while the H term arises from the interaction of opposing poles. Clearly the variance becomes infinite on account of the Fejér kernel terms, while the cross-term H remains bounded asymptotes to a constant.

The behavior of the terms in (153) as a function of t and ω for the particular choice $\omega_0 = 1$ are shown in Fig. 3. The full $|\widetilde{\mathsf{C}}|^2$ term is partitioned into a term containing two Fejér kernels, with their smooth behavior as t increases, plus the "residual" H term. The latter contains chains of opposing extrema which shrink in extent and increase in magnitude towards large times. To find the complete behavior of the variance, one can visualize multiplying the pattern of $|\widetilde{\mathsf{C}}|^2$ shown in Fig. 3a by $S(\omega)$ and integrating "horizontally", leaving a "vertical" profile which gives the variance for all times. If the spectrum consists of a delta-function at ω_1 say, then the variance as a function of time is given by a "slice" through this pattern at $\omega = \omega_1$. At several choices of a fixed frequencies ω_1, one has the behavior shown in Fig. 4. Again one sees the complete $|\widetilde{\mathsf{C}}|^2$ kernel is partitioned into the smooth Fejér term including a quadratic growth at resonance, plus an oscillatory bounded oscillatory term. Note that the quadratic growth of the Fejér term occurs because of the multiplication of F by t in (153).

The variance may therefore be approximated, neglecting the asymptotically bounded H term, by

$$\sigma_y^2(t) \approx \frac{1}{(2\omega_0)^2}[t\widetilde{S}(-\omega_0,t) + t\widetilde{S}(\omega_0,t)] \tag{156}$$

where $\widetilde{S}(\omega_0,t) = \widetilde{S}_t(\omega)$, the periodogram at time t, is defined by (10) with t replacing T, and with the corresponding rate of change of variance being

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_y^2(t) \approx \frac{1}{(2\omega_0)^2}(\mathsf{D}_{2t} * S[\omega-\omega_0] + \mathsf{D}_{2t} * S[\omega+\omega_0]) \tag{157}$$

using (53). The variance under this approximation at time t is just proportional to t times the periodogram at that instant, evaluated at plus or minus the natural frequency. Similarly, the rate of change of variance is proportional to two quantities at $\pm\omega_0$ which are similar to the periodogram, but which involve the smoothing kernel $\mathsf{D}(\omega,2t) = \Re\widetilde{\mathsf{D}}(\omega,t)$ instead of the Fejér kernel. Thus one may say that the system is always responding instantaneously to the forcing which appears to be present at the natural frequencies. When $S(\omega)$ is continuous at $\pm\omega_0$, (157) asymptotes to result of [3], (9), as a consequence of the asymptotic behavior of the Dirichlet kernel (41).

Additional insight into the behavior of the variance may be found by writing (157) directly as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_y^2(t) \approx \frac{1}{(2\omega_0)^2}[\widetilde{S}(-\omega_0,t) + \widetilde{S}(\omega_0,t)] + \frac{1}{(2\omega_0)^2}\left[t\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{S}(-\omega_0,t) + t\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{S}(\omega_0,t)\right]. \tag{158}$$

If the spectrum is locally white, by which we mean that it is "flat" in some suitable chosen region surrounding $\pm\omega_0$, then $\mathrm{d}\widetilde{S}/\mathrm{d}t \approx 0$ and one may neglect the second bracketed term in (158). In this case the variance differs from its limiting value only in the replacement of the true spectrum by the apparent spectrum. Thus one may say that limit (9) emerges when the spectrum appears locally white in the vicinity of the natural frequency, which is expected to be true at some time provided the spectrum is continuous at $\pm\omega_0$.

In Section 3.4, we stated that the linear growth of the variance of $X_T(\omega)$ is equivalent to that in the random-walk problem. The same is true for the linear growth of variance for the oscillator when the spectrum near $\pm\omega_0$ is locally white. Indeed,

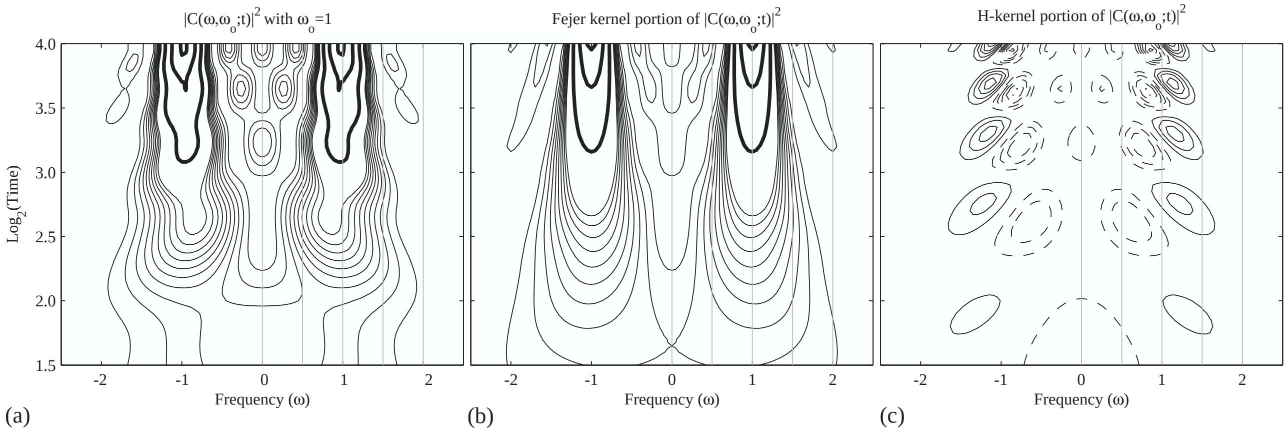

Fig. 3. The $|\widetilde{\mathsf{C}}|^2$ kernel (a) and its partitioning into the sum of two Fejér kernel terms (b) plus a residual "H" term (c). Contour are: thin solid lines, [1:1:10]; thin dashed lines, -[1:1:10]; and thick solid lines, [20:20:100]. Vertical lines mark fixed frequencies to be referred to in the next figure.

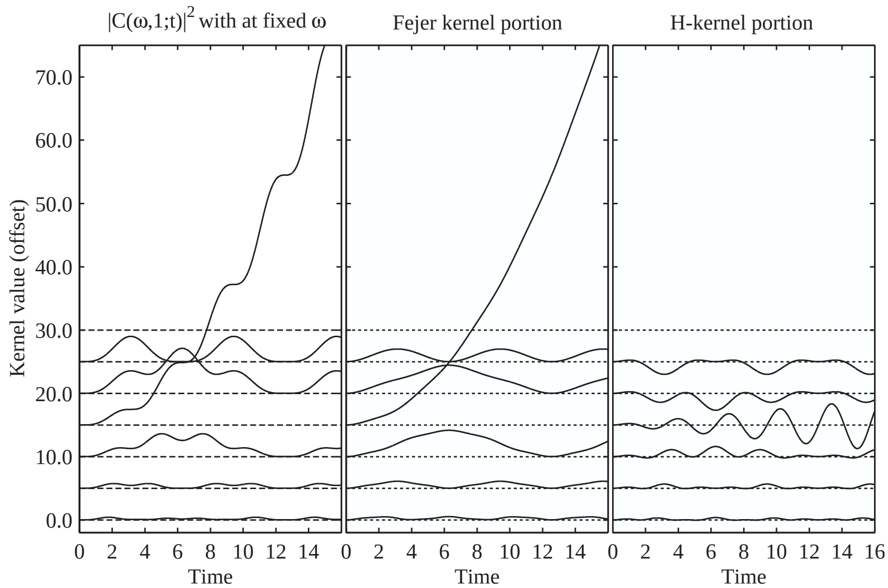

Fig. 4. Profiles of the patterns shown in Fig. 3 as a function of time, along the frequencies indicated in that figure. The thick solid curve in all panels is for the resonant frequency $\omega = \omega_0$, while the dashed curve is for $\omega = 0$. Curves from successively lower frequencies are offset upwards, with dotted lines showing the locations of the zero baselines.

since the solution (146) depends only on the apparent spectral process at $\pm\omega_0$, the growing variance of the oscillator may be traced back to the growing variance of the $\widetilde{X}_T$ at plus or minus the natural frequency.

On the other hand if the forcing consists of a delta function at ω_0

$$S(\omega) = 2\pi\,\delta(\omega - \omega_0)$$

then $\widetilde{S}(\omega_0, t) = t$ and both bracketed terms in (158) contribute equally, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma_y^2(t) = \frac{1}{2\omega_0^2}\,t \tag{159}$$

in agreement with that directly computed for the resonantly forced solution $y_1(t;\omega_0,\omega_0)$ (17). More generally, for an arbitrary sinusoidal forcing

$$S(\omega) = 2\pi\, \delta(\omega - \omega_1),$$

the rate of change of variance is

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} E\{|y_1(t;\omega_1,\omega_0)|^2\} = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} |\widetilde{\mathsf{C}}(\omega,t;\omega_0)|^2 = \frac{\pi}{2\omega_0^2}\,[\delta(\omega_1+\omega_0) - \delta(\omega_1-\omega_0)] \tag{160}$$

recovering the second result of [3], our (14).

In Section 2, we mentioned that it is unclear what is meant by the "resonant term" in the case of spectrally continuous random forcing. Here we suggest that it is more useful to speak of linear versus quadratic growth. The case of quadratic growth corresponds to the system forced by a discrete mode at the resonant frequency, and therefore generalizes the concept of resonance from the discrete-mode case. However, as shown above, an initial phase of quadratic growth is to be expected for all continuous spectra, because initially the transfer kernel is infinitely broad and hence all (realistic) spectra will initially appear to contain delta-functions at the resonant frequency. At sufficiently long times, the variance will transition to linear growth, at which point the spectrum appears flat and so the transfer kernel samples a stable value.

Note that had we made the discrete-mode approximation first by projecting onto Fourier modes over a certain time interval, and then looked at the "resonant" term, we would have found as in (17) that the variance grows quadratically for all times. Therefore this framework does not permit the distinction between the quadratic and linear growth phases. Further, performing our projection over a new time extended time interval, we would obtain a different "resonant term" which would in general not agree with the earlier result. This leads to the paradoxical conclusion that the "resonant term" depends on one's choice of reference frame. Taking the expected value will of course recover the correct rate of change of variance, but this is rather deeply buried within the discrete-mode model. As shown in (157), the Fourier components of the forcing associated with instantaneous growth of variance evolve with time. Therefore the concept of a "resonant term" seems inappropriate for the case of ongoing, spectrally continuous forcing, and the approximation (17) is not useful. Instead, we suggest approximating the variance as (156), and using the distinction between quadratic growth and linear growth to qualitatively describe how the oscillator is "perceiving" the forcing.

5.4. Comments on one-sided oscillators

Some oscillating systems satisfy the "one-sided" oscillator equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\omega_0\right) y[t] = x(t), \tag{161}$$

for example, inertial waves in the ocean surface mixed layer. In the latter case ω_0 is denoted f and is known as the Coriolis frequency

$$f = 2\Omega \sin(\phi),$$

where ϕ is the latitude and Ω is the earth's angular frequency of rotation. For such a system, the statistical behavior is even simpler. The variance is simply t times the apparent spectrum at the resonant frequency. It follows that the variance divided by time for an array of such systems with varying ω_0 is an exact physical analogue to computing the periodogram. Thus, one may imagine the periodogram to be computed by using the input signal to force inertial waves in the ocean surface mixed layer at a range of latitudes. Furthermore, for such a one-sided oscillator, one may shift the input spectral process through a multiplication by $\mathrm{e}^{-\mathrm{i}\omega_1 t}$, so the spectrum at any frequency may be determined through a suitable premultiplication. Therefore a single (frictionless) one-sided oscillator can in principle be used to compute the periodogram.

5.5. Forcing by a "wavenumber" grid

As an example of a physical application, consider the problem in which the forcing is distributed over a "wavenumber" plane; this problem arises in nonlinear wave triad interactions. Let the forcing be represented by

$$x(t) = \frac{1}{L^2} \sum_n A(\mathbf{k}_n)\, \mathrm{e}^{\mathrm{i}\Omega_*(\mathbf{k})\, t}, \tag{162}$$

where the sum is over some discrete grid of wavenumbers indexed by a parameter n, $\Omega_*(\mathbf{k})$ is some function relating wavenumber to frequency, and L is the length of the corresponding box in (x, y) space. One may view the sum on the right-hand side as a discrete approximation to a wavenumber integral, or as a nonuniformly spaced frequency-domain Fourier transform as in (15). Taking the latter perspective $X(\omega)$ is found to be

$$X(\omega) = \frac{2\pi}{L^2} \sum_n A(\mathbf{k}_n)\, \delta(\omega - \Omega_*(\mathbf{k})) \tag{163}$$

with the solution to the simple harmonic oscillator being

$$y(t) = \frac{\Gamma(t)}{L^2} \sum_n A(\mathbf{k}_n) \widetilde{\mathrm{C}}(\Omega_*(\mathbf{k}_n), t; \omega_0)\, \mathrm{e}^{\mathrm{i}\Omega_*(\mathbf{k})\, t}, \tag{164}$$

which is known for all $t > 0$ in terms of the $A(\mathbf{k}_n)$. The non-uniformly spaced Fourier representation allows us to exploit the remarkable power of the delta function to obtain the simple solution (164), without the need to make a change of variables.

As we let L approach infinity, all summations become integrals and the solution is

$$y(t) = \frac{\Gamma(t)}{(2\pi)^2} \iint A(\mathbf{k})\, \widetilde{\mathrm{C}}(\Omega_*(\mathbf{k}_n), t; \omega_0) \times \mathrm{e}^{\mathrm{i}\Omega_*(\mathbf{k})\, t}\, \mathrm{d}\mathbf{k}. \tag{165}$$

Assume that $\Omega_*(\mathbf{k}_n) \pm \omega_0$ vanishes along certain curves on the wavenumber plane, called the "resonance curves". Denote by $\lambda_m^+(k)$ the M curves such that $\Omega_*(\mathbf{k}_n) - \omega_0 = 0$, and by $\lambda_n^-(k)$ the N curves such that $\Omega_*(\mathbf{k}_n) + \omega_0 = 0$. Geometrically (165) states that we place $\widetilde{\mathrm{D}}(\omega, t)$ functions along these resonance curves, so that at each resonance curve one has the $T/(2\pi)$ behavior and moving outward from the curve one obtains a profile of $\widetilde{\mathrm{D}}(\omega)$. Obviously (165) will asymptote to delta-functions along these resonance curves plus a "distributed" response which will have a $1/\omega$-type discontinuity across them.

To find the asymptotic response, we use

$$\delta(g(x)) = \sum_n \frac{\delta(x - x_n)}{g'(x)}, \tag{166}$$

(see [1] Eq. (1.180)) where the sum is over all roots $g(x_n) = 0$ for which the first derivative of g does not vanish ($g'(x_n) \neq 0$). The asymptotic solution may be readily found to be

$$y(t) = \frac{1}{2\mathrm{i}\omega_0} \frac{\Gamma(t)}{4\pi} \times \left[\mathrm{e}^{\mathrm{i}\omega_0 t} \sum_{m=1}^{M} \int \frac{A(k, \lambda_m^+)}{\Omega'_*(k, \lambda_m^+) \sin(\Phi_m^+(k))}\, \mathrm{d}k - \mathrm{e}^{-\mathrm{i}\omega_0 t} \sum_{n=1}^{N} \int \frac{A(k, \lambda_n^-)}{\Omega'_*(k, \lambda_n^-) \sin(\Phi_n^-(k))}\, \mathrm{d}k \right] - \frac{1}{(2\pi)^2} \iint \frac{A(\mathbf{k})}{\Omega_*^2(\mathbf{k}) - \omega_0^2}\, \mathrm{e}^{\mathrm{i}\Omega_*(\mathbf{k})\, t}\, \mathrm{d}\mathbf{k}, \tag{167}$$

where $\Phi_n^{\pm}(k) \equiv \arctan(\lambda_n^{\pm}(k)/k)$. Note the similarity with (146), the "variation of parameters" formulation, except that in the above we have separated the "local" and "distributed" terms and combined the latter to form the third term in (167). The local and distributed responses are again 90° out of phase. To compute the local response, one need only integrate along resonance curves, while the distributed responses requires an integration over the plane.

6. Conclusions

The response of a linear time-invariant operator to stochastic forcing, which is switched on at a certain time and allowed to continue indefinitely, is considered. Absorbing the "truncation" of the forcing into the operator, one obtains Fourier domain convolutions between the forcing and well-known smoothing kernels. The quantity controlling the solution is a time-dependent "transfer kernel", formed by a simple smoothing of the transfer function. The resulting "moving endpoint" solution may be expressed in a way which is formally identical to the solution of the corresponding time-invariant problem, with the transfer kernel substituted for the transfer function. Thus the moving endpoint solution may be considered to generalize the time-invariant solution to the nontime-invariant switch-on problem.

In the process of deriving this solution, we develop a new mathematical framework which includes an "operator" notation, an extensive use of the Hilbert transform, and an emphasis on kernel functions and their properties. Revisiting standard material on Fourier theory and linear time-invariant operators with this new approach, we create a foundation which then allows the switch-on problem to be solved with ease.

This method is used to solve the switch-on problem for the simple harmonic oscillator. In this case the smoothed transfer function consists of two "rotated" (i.e. complex-valued) Dirichlet kernels, centered at plus or minus the resonant frequency. The asymptotic behavior of the simple harmonic oscillator is investigated. While some of the asymptotic expressions have been presented earlier by [3] and elsewhere, the moving endpoint approach provides new insight into the underlying physics. We find the asymptotic behavior of the oscillator is essentially identical to its instantaneous behavior, provided the "true" spectrum appearing in the asymptotic expressions is replaced by the "apparent" spectrum, i.e. the instantaneous periodogram. In both cases the variance grows linearly with the apparent value of the spectrum at the resonant frequency. The asymptotic response differs for a purely continuous forcing compared with a resonant "line" forcing. In the former case, the growth of variance is linear with time; in the latter case, it is quadratic.

The asymptotic response for continuous forcing is equivalent to the response for a spectrally white forcing. Thus one may say that the asymptotic response arises when the process is sampled well enough that it appears spectrally white, so that the apparent spectrum at the resonant frequency becomes stable. Then the growth of variance of the response may be ascribed to the growth of variance of the apparent spectral process, which in turn is equivalent to a random walk. This provides a simple physical interpretation of the linear growth of variance with time. However, the initial growth of the response to spectrally continuous forcing will also be quadratic. This occurs because for sufficiently short times after switch-on, any forcing will appear to contain delta-functions at the resonant frequencies. As time increases, and the smoothing of the spectral process becomes less severe, the growth rate of variance becomes linear.

The uncertainty about the spectral process, introduced by the smoothing effect of truncation, is fundamental in determining the response of the oscillator. The oscillator encounters the same blurring of information that occurs when one attempts to estimate the spectrum of a signal from a finite sample.

Particularly simple expressions are found when the forcing consists of a series of sinusoids, whether or not these are located at equally spaced frequencies. Further, the generalization to the case of a forcing distributed over a (wavenumber) plane is straightforward. These expressions in particular should be of practical value in computations. In a succeeding paper, the results of this work are applied to the investigation of the bispectrum arising from resonantly interacting gravity-capillary waves.

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